

Calculus III: Coursework 2
M.A.H. MacCallum 2007

The deadline is 4PM on Tuesday Oct 9th. Hand-in is to the green box on the ground floor of the Maths building.

1. Find the gradient (∇f) for $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$. [10]

Answer

$$\nabla f = -6xz\mathbf{i} - 6yz\mathbf{j} + (6z^2 - 3(x^2 + y^2))\mathbf{k}.$$

Marks: 4 for method, 2 per correct component. No marks for an answer which is not a vector.

2. Prove that $\nabla(r^n) = nr^{n-2}\mathbf{r}$, where $r = (x^2 + y^2 + z^2)^{1/2}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. [15]

Answer

$$r = \sqrt{x^2 + y^2 + z^2} \text{ so } \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}. \text{ Thus by the chain rule, for } f = r^n,$$

$$\frac{\partial f}{\partial x} = nr^{n-1} \frac{\partial r}{\partial x} = nr^{n-2}x$$

The results for y and z are similar so we get

$$\nabla f = nr^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nr^{n-2}\mathbf{r}.$$

(Note it can also be done directly from $r^n = (x^2 + y^2 + z^2)^{n/2}$ by applying the chain rule and identifying the result.)

Marks: Use of chain rule 5, a correct evaluation of gradient 5, identification with result required 5.

3. Consider a sphere of radius 3 centred at the origin.

- (a) Express the surface of the sphere in the form $f(x, y, z) = 0$.
- (b) Find a plane tangent to the sphere at the point $\mathbf{r}_1 = (1, 2, 2)$. [20]
- (c) Find the position vector of points on the line through $\mathbf{r}_1$ and $\mathbf{r}_2 = (2, 1, 2)$.
- (d) Find the position vector of points on a line through $\mathbf{r}_1$ and normal to the sphere.

Answer

We have

(a) $x^2 + y^2 + z^2 - 9 = 0$

(b) Taking the gradient at $\mathbf{r}_1$ gives a vector normal to the surface

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

The tangent plane is $(\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{p} = 0$. Thus

$$[(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z-2)\mathbf{k}] \cdot [\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}] = 0$$

or

$$x + 2y + 2z = 9.$$

Alternatively, use the fact that writing a surface as $z = f(x, y)$ the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = (x - x_0) \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} + (y - y_0) \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)},$$

which was done in Calculus II.

(c) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} - \mathbf{j})$ where $-\infty < s < \infty$.

(d) $\mathbf{r} = s(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ where $-\infty < s < \infty$.

(Normal direction can be inferred from the answer to (b) if (b) was not calculated that way.)

Marks: Parts a–d (4 each) for any correct version (e.g. in row or column vector form, or with other parameters), plus another 4 for correct gradient or partial derivatives in part (b). Note many answers may give $2\mathbf{p}$ for the normal and correctly use that in parts b and d, and may give $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + s(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ in part d. Accept these as correct answers but note the results could be simpler.

4. Find equations for the (i) tangent plane and (ii) normal line at the point P_0 on the surface $x^2 + y^2 - z^2 = 18$, where $P_0 = (3, 5, -4)$. [20]

Answer

$\nabla f = (2x, 2y, -2z) = (6, 10, 8)$ at $\mathbf{P}_0$.

A point $\mathbf{r}$ in the tangent plane has $(\mathbf{r} - \mathbf{P}_0) \cdot \nabla f = 0$, i.e.

$$\begin{aligned} (x-3, y-5, z+4) \cdot (6, 10, 8) &= 6(x-3) + 10(y-5) + 8(z+4) = 0, \\ \Rightarrow 6x + 10y + 8z &= 18 + 50 - 32 = 36 \\ \Rightarrow 3x + 5y + 4z &= 18 \end{aligned}$$

Normal line is $\mathbf{r} = \mathbf{P}_0 + t\nabla f$ or

$$\mathbf{r} = (3 + 6t)\mathbf{i} + (5 + 10t)\mathbf{j} + (-4 + 8t)\mathbf{k}.$$

or it can be written

$$(\mathbf{r} - \mathbf{P}_0) \times \nabla f = (4y - 5z - 40)\mathbf{i} + (3z - 4x + 24)\mathbf{j} + (5x - 3y)\mathbf{k} = \mathbf{0}$$

Marks: Each part 4 method, 6 accuracy.

5. Find the derivative of the scalar field $V = x^2yz + 4xz^2$ in the direction of the vector $(2, -1, -1)$ at the point $P = (1, -2, -1)$. [Hint: write down the unit vector $\mathbf{n}$ which has the same direction as the given vector, and evaluate $\mathbf{n} \cdot \nabla V$ at the point P .] [20]

Answer

The vector $(2, -1, -1)$ has length $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$, so the unit vector is $(2, -1, -1)/\sqrt{6}$.

$$\nabla V = (2xyz + 4z^2)\mathbf{i} + x^2z\mathbf{j} + (x^2y + 8zx)\mathbf{k}.$$

which is $(8, -1, -10)$ at P . Hence $\mathbf{n} \cdot \nabla V = (2, -1, -1) \cdot (8, -1, -10)/\sqrt{6} = 27/\sqrt{6}$.

Marks: Finding $\mathbf{n}$: 3 for method, 3 for answer. Gradient at a general point 4, substituting P 3. Knowing how to evaluate dot product 3, final answer 3.

6. If $f = xyz$, evaluate $\int_V f \, dV$ where

- (a) V is the rectangular volume bounded by the planes

$$x = 0, a, \quad y = 0, b, \quad z = 0, c,$$

- (b) V is the rectangular volume bounded by the planes

[15]

$$x = \pm a, \quad y = \pm b, \quad z = \pm c.$$

Answer

- (a)

$$\int_0^a \int_0^b \int_0^c xyz \, dz \, dy \, dx = a^2 b^2 c^2 / 8.$$

- (b)

$$\int_{-a}^a \int_{-b}^b \int_{-c}^c xyz \, dz \, dy \, dx = 0.$$

Marks: Method correct in at least one part 5, accuracy 5 per part.